Problems for the lectures by Prof. V. Mukhanov

Balkan Summer Institute 2011,

Cosmology and Particle Physics Beyond the Standard Models

August 24, 2011

1 Homogeneous and Isotropic Metric

By embedding a three-dimensional sphere (pseudo-sphere) in a four-dimensional Euclidean (Lorentzian) space, verify that the metric of a three- dimensional space of constant curvature can be written as

$$\mathrm{d}l_{3\mathrm{d}}^2 = a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)\right).$$

2 Redshift

Light wave was emitted with a frequency ω_0 when the universe had scale factor a_0. Find the frequency of these signal ω received by an observer at some galaxy which is located not very far away from the emitter when the universe had the scale factor a. How does the temperature of a relativistic very hot gas scale with the size of the universe a? How does the temperature of a gas of massive nonrelativistic particles scale?

3 Friedmann equations

Obtain the first Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\varepsilon,$$

where $H = \dot{a}/a$ and the Hubble parameter and $k = const$ from the continuity equation

$$\dot{\varepsilon} = -3H(\varepsilon + p),$$

and the "second law"

$$\ddot{a} = -\frac{4\pi G}{3}(\varepsilon + 3p)a.$$

4 Energy Momentum Tensor for scalar field I

Consider the conservation of the Energy Momentum Tensor (EMT)

$$T^{\alpha}_{\beta,\alpha} = 0\,,$$

for a scalar filed

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - g_{\alpha\beta}\left(\frac{1}{2}\phi_{,\lambda}\phi^{,\lambda} - V\left(\phi\right)\right)\,, \tag{1}$$

and derive the Klein-Gordon equation.

5 Energy Momentum Tensor for scalar field II

Rewrite EMT (1) in form of the EMT for the perfect fluid:

$$T_{\alpha\beta} = \left(\varepsilon + p\right) u_{\alpha}u_{\beta} - pg_{\alpha\beta}\,,$$

where u^{λ} is a unit timelike and is future directed vector of the fluid 4–velocity. Does the energy density ε transform under Lorentz transformation? Does the equation of state $p\left(\varepsilon\right)$ exist? What is equation of state parameter $w = p/\varepsilon$ sometimes loosely called just "equation of state"?

6 Oscillations of the inflaton

Consider a homogenous scalar field $\phi\left(t\right)$ with the potential $V\left(\phi\right)$ and a canonical kinetic term. Assume that $\phi\left(t\right)$ oscillates with the period much shorter than H^{-1}. What is the effective action of state $w = p/\varepsilon$ relevant for the cosmological evolution on time scales H^{-1}? Specialize for the potentials with the asymptotic behavior

1. $V \sim \phi^{n}$ in particular with $n = 2$ and $n = 4$
2. $V \sim \ln\left(\phi/\phi_{\mathrm{c}}\right)^{2}$.