Balkan School 2011
Problems 1

Prepared by Goran and Miha

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1. **Dirac spinor transformations.** A four-component Dirac spinor transforms under the Lorentz group as

$$\psi \to S\,\psi,\; S = e^{\frac{i}{2}\theta_{\mu\nu}\Sigma^{\mu\nu}},\; \Sigma^{\mu\nu} \equiv \frac{1}{4i}\left[\gamma^\mu, \gamma^\nu\right], \tag{1}$$

where γ^μ obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2}$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},\; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{3}$$

   (a) Introduce left and right-handed chiral spinors

$$\psi_{L,R} \equiv \frac{1 \pm \gamma_5}{2}\psi,\; \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

   Show that

$$u_{L(R)} \to e^{-i\sigma/2(\theta \mp i\phi)} u_{L(R)}, \tag{5}$$

   with

$$\psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ u_R \end{pmatrix}. \tag{6}$$

   What are θ and ϕ?

   (b) Take a boost in the z direction and find an expression for ϕ.

2. **Parity and charge conjugation.** Define a charge conjugate spinor

$$\psi^c \equiv C\overline{\psi}^T \tag{7}$$

with

$$C^T\gamma^\mu C = -\gamma_\mu^T,\; C^T = -C,\; C^\dagger = -C, \tag{8}$$

(we chose explicitly $C = i\gamma_2\gamma_0$).

   (a) Show that

$$\psi^c \to S\,\psi^c \tag{9}$$

   under the Lorentz group, which shows that ψ^c transforms the same way as ψ, i.e. it is also a proper spinor.

   (b) Take ψ_L and compute its charge conjugate. What is its chirality?

(c) What happens to u_L and u_R under a parity transformation? Recall that

$$\mathcal{P} : \psi \to \gamma_0 \, \psi. \tag{10}$$

3. **Majorana fermions:** Take a left-handed Weyl field

$$\psi_L = \begin{pmatrix} u_L \\ 0 \end{pmatrix} \tag{11}$$

and construct the following Majorana field

$$\psi_M = \psi_L + C\overline{\psi}_L^{\,T} \tag{12}$$

(a) Show that

$$\psi_L^T C \psi_L \tag{13}$$

is Lorentz invariant.

(b) See that

$$\overline{\psi}_M \gamma^\mu \partial_\mu \psi_M = 2\,\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L, \tag{14}$$

and that the Majorana mass terms can be rewritten as

$$\overline{\psi}_M \psi_M = \psi_L^T C \psi_L + \text{h.c.}. \tag{15}$$